Synchronization performance of complex oscillator networks

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(Received 19 December 2008; revised manuscript received 18 September 2009; published 30 November 2009)

Recently, synchronization of complex networks has attracted increasing attention from various research fields. However, most previous works focused on the stability of synchronization manifold. In this paper, we analyze the time-delay tolerance and converging speed of synchronization. Our theoretical analysis and extensive simulations show that the critical value of time delay for network synchronization is inversely proportional to the largest Laplacian eigenvalue, the converging speed without time delay is proportional to the second least Laplacian eigenvalue, and the time delay could increase the converging speed linearly for heterogeneous networks and significantly for homogeneous networks.

DOI: 10.1103/PhysRevE.80.056116 PACS number(s): 89.75.−k, 05.45.Xt, 89.20.Ff

Synchronization of interconnected systems has attracted considerable research interests among physics, biology, and engineering communities in recent years [1–4]. Many interconnected systems can be represented as complex networks [5–7], in which the nodes are oscillators (dynamical systems) and the edges are the coupling among them. Most previous works focused on the stability of the synchronized state (so-called synchronizability) of various complex oscillator networks, i.e., to demonstrate the conditions under which the oscillator networks can become synchronized and, furthermore, to devise some efficient and effective methods to improve the synchronizability. The first milestone in this direction is the work of Pecora and Carroll [8], which unravelled a key synchronized condition by using the master stability function (MSF) described as follows. Model the dynamics of N coupled identical oscillators by $\dot{\mathbf{x}}_i=\mathbf{F}(\mathbf{x}_i)-c\sum_{j=1}^{N} l_{ij}\mathbf{H}(\mathbf{x}_j)$, where $\mathbf{x}_i$ is the m-dimensional state of node i, $i=1,2,\ldots,N$, $\mathbf{F}$ describes the individual node evolution of the states without coupling, c is the constant coupling strength, $\mathbf{H}$: $\mathbb{R}^m \to \mathbb{R}^m$ is the coupling function, and $\mathbf{L}=(l_{ij})_{N\times N}$ is the Laplacian matrix representing the network structure defined by $l_{ii}=k_i$ (the degree of node i), $l_{ij}=-1$ if node i and j are connected but $l_{ij}=0$ otherwise. One can linearize the coupled dynamical systems around the synchronized state $\mathbf{x}_1=\mathbf{x}_2=\ldots=\mathbf{x}_N=\mathbf{x}^s$, typically, $\dot{\mathbf{x}}^s=\mathbf{F}(\mathbf{x}^s)$, and then diagonalize $\mathbf{L}$ to find its N eigenvalues $0=\lambda_1<\lambda_2\leq\ldots\leq\lambda_N=\lambda_{\max}$. This yields the block variational equations $\dot{\xi}_h=[D\mathbf{F}-c\lambda_h D\mathbf{H}]\xi_h$, where λ_h is the hth eigenvalue of $\mathbf{L}$, $h=1,2,\ldots,N$, and $D\mathbf{F}$ and $D\mathbf{H}$ are the Jacobian matrices of $\mathbf{F}$ and $\mathbf{H}$, respectively, which are the same for each block. Therefore, if one wants to study the synchronization properties with respect to different undirected networks, one could just compute the maximum Lyapunov exponents $\beta_{\max}^h$ of the above equations as functions of $c\lambda_h$. If the values of $\beta_{\max}^h$ are all negative, the synchronized state $\mathbf{x}^s$ is stable. Moreover, in [8], it is observed that the maximum Lyapunov exponents $\beta_{\max}^h$ are in general negative only within an interval (α_a,α_b), that is, $\alpha_a<c\lambda_2\leq\ldots\leq c\lambda_{\max}<\alpha_b$. When $\alpha_b<\infty$, the synchronization of the network is stable if $\frac{\lambda_{\max}}{\lambda_2}<\frac{\alpha_b}{\alpha_a}$. Thus, one can reduce the value of the eigenratio $\frac{\lambda_{\max}}{\lambda_2}$ to improve the synchronizability of the network. It is noteworthy that in some cases, such as $\mathbf{H}(\mathbf{x})=\mathbf{x}$ and $\alpha_b=\infty$, the synchronized state is stable when the coupling strength $c>\frac{\alpha_a}{\lambda_2}$ [9,10].

On one hand, many papers have followed the MSF method and studied the synchronizability of unweighted [11–18] or weighted [19–21] networks. Several approaches have been proposed to improve the synchronizability, such as static weighted coupling based on edge betweenness [22], topology modification [23,24], optimization [25–27], adaptive evolution [28], and so on. On the other hand, there are few works studying the synchronizing processes. Among others [29], studied the Kuramoto model on hierachical networks and revealed a gradient synchronization process. The results of [30] showed the evolution pattern of synchronization. And we found [31] that even the same value of the eigenratio $\frac{\lambda_{\max}}{\lambda_2}$ could induce different performances of the synchronizing process. Therefore, it is important to further investigate the performances, e.g., converging speed and time-delay tolerance of synchronizing processes.

In this paper, we study analytically and numerically the time-delay tolerance and the converging speed of the synchronizing processes of complex oscillator networks. Our results show, under some mild approximations, that (i) the critical time-delay tolerance τ_c is inversely proportional to $c\lambda_{\max}$, (ii) the converging speed without time delay is proportional to $c\lambda_2$, and (iii) the converging speed with time-delay τ ($<\tau_c$) increases with $c\lambda_2$ and also increases with τ. Moreover, the time-delay τ could increase the converging speed linearly for heterogeneous networks but could increase it significantly for homogeneous networks.

We choose the coupling function $\mathbf{H}(\mathbf{x})=\mathbf{x}$ to perform theoretical analysis as follows. Adding time delay into the network gives

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$$\dot{\mathbf{x}}_i(t) = \mathbf{F}[\mathbf{x}_i(t)] - c\sum_{j=1}^{N} l_{ij}\mathbf{x}_j(t-\tau), \tag{1}$$

where τ is the time-delay constant. It is noteworthy that the time-delay format we adopted here is employed in many real-world applications [32,33]. For unweighted and undirected networks, let the synchronization errors be $\xi = (\xi_i, \xi_2, \ldots, \xi_N) = (\mathbf{x}_1 - \mathbf{x}^s, \mathbf{x}_2 - \mathbf{x}^s, \ldots, \mathbf{x}_N - \mathbf{x}^s)$. Then, one can linearize Eq. (1) as

$$\dot{\xi}(t) = D\mathbf{F}\xi(t) - c\xi(t-\tau)\mathbf{L}. \tag{2}$$

Denote by $\mathbf{e}_h$ the eigenvectors associated with the eigenvalues λ_h of $\mathbf{L}$ and then multiply it to both sides of Eq. (2), one obtains $\dot{\xi}(t)\mathbf{e}_h = D\mathbf{F}\xi(t)\mathbf{e}_h - c\lambda_h\xi(t-\tau)\mathbf{e}_h, h=1,2,\ldots,N$. To deal with the time-delay factor, apply the first-order approximation $\xi(t-\tau) = \xi(t) - \tau\dot{\xi}(t)$ for small τ, one obtains

$$\dot{\eta}_h(t) = \frac{1}{1 - c\tau\lambda_h}(D\mathbf{F} - c\lambda_h)\eta_h(t), \tag{3}$$

where $\eta_h(t) = \xi(t)\mathbf{e}_h$. Ref. [8] has shown that the dynamical process without time-delay ($\tau=0$) is linearly stable, i.e., the maximum Lyaponov exponents are all negative if $c\lambda_h \in (\alpha_a, \alpha_b)$ for all h $(h \neq 1)$. Thus, the dynamical process with time delay, as described by Eq. (3), is linearly stable if $1 - c\tau\lambda_h > 0$ for all h $(h \neq 1)$, which induces the critical time-delay tolerance

$$\tau_c = \min_{h\neq 1}\frac{1}{c\lambda_h} = \frac{1}{c\lambda_{\max}}. \tag{4}$$

In order to provide a clear figure about the converging speed approaching the synchronized state, consider the parameters $W_h(t) = \|\eta_h\|^2 = \eta_h^T\eta_h$ associated with λ_h. From Eq. (3), one can easily get that

$$\frac{\dot{W}_h(t)}{W_h(t)} = \frac{1}{1 - c\tau\lambda_h}\frac{\eta_h^T(D\mathbf{F} + (D\mathbf{F})^T)\eta_h}{\eta_h^T\eta_h} - \frac{2c\lambda_h}{1 - c\tau\lambda_h}. \tag{5}$$

Because the focus here is on the asymptotical converging speed of ensemble oscillators caused by the coupling and $D\mathbf{F}$ is an essential dynamical property of the individual uncoupled oscillators, under the assumption $\|D\mathbf{F}\| < \infty$, we consider $D\mathbf{F} + (D\mathbf{F})^T$ as fluctuation of $\eta_h^T\eta_h$. To justify its validity, we compute the fluctuation factors

$$\zeta_h(t) = \frac{\eta_h^T(D\mathbf{F} + (D\mathbf{F})^T)\eta_h}{\eta_h^T\eta_h}, \tag{6}$$

in the right side of Eq. (5) directly. Figures 1(a) and 1(b) display the typical evolution of the values of $\zeta_2(t)$ and $\zeta_N(t)$ corresponding to the second least (nonzero) eigenvalue λ_2 and the largest eigenvalue λ_N, for the Rössler oscillator [34]. One can see that the values are fluctuated around zero. We have also computed the fluctuation factors for Lorenz chaotic oscillator [35] and obtained similar results (not shown). Thus, one can rewrite Eq. (5) as

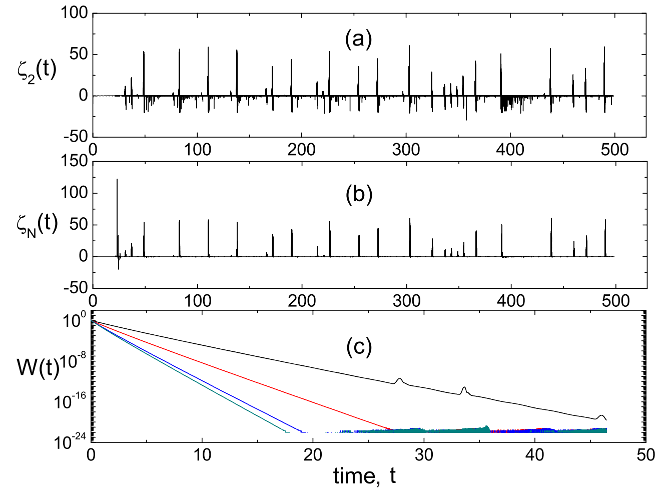

FIG. 1. (Color online) The values of the fluctuation factors $\zeta_2(t)$ and $\zeta_N(t)$ are displayed in (a) and (b) for Rössler oscillator on the network with that the size $N=1500$, the average degree $\langle k\rangle=40$, $\delta=1.0$, the time-delay $\tau=0.05$ (see the latter text for details). (c) $W(t) = \Sigma_i|\mathbf{x}_i(t) - \langle\mathbf{x}_i\rangle|^2$ as a function of the evolution time t for different networks with the same size $N=1500$. From right to left, the curves present (i) $\langle k\rangle=20$, $\delta=0.0$, $\tau=0$, (II) $\langle k\rangle=30$, $\delta=0.5$, $\tau=0$, (III) $\langle k\rangle=40$, $\delta=1.0$, $\tau=0$, and (IV) $\langle k\rangle=40$, $\delta=1.0$, $\tau=0.05$, respectively. The coupling strength $c=0.08$.

$$\dot{W}_h(t) = \frac{\zeta_h(t) - 2c\lambda_h}{1 - c\tau\lambda_h}W_h(t), \tag{7}$$

where the function $\zeta_h(t)$ denotes the fluctuation of $\eta_h^T\eta_h$ caused by $D\mathbf{F} + (D\mathbf{F})^T$. The solution is approximately $W_h(t) = W_h(0)e^{\zeta_h(t) - 2c\lambda_h/1 - c\tau\lambda_h t}$. In other words, by ignoring the fluctuation, the converging speed approaching exponentially the synchronized state along the eigenvector $\mathbf{e}_h$ is $\mu_h(\tau) = -\frac{d\ln W_h(t)}{dt} = \frac{2c\lambda_h}{1 - c\tau\lambda_h}$. Therefore, as the converging speed is restrained by the slowest mode, the converging speed of the synchronizing process without time delay is

$$\mu(0) = \min_{h\neq 1} 2c\lambda_h = 2c\lambda_2, \tag{8}$$

while the converging speed of the synchronizing process with time-delay τ $(<\tau_c)$ is

$$\mu(\tau) = \min_{h\neq 1}\frac{2c\lambda_h}{1 - c\tau\lambda_h} = \frac{2c\lambda_2}{1 - c\tau\lambda_2} = \frac{\mu(0)}{1 - c\tau\lambda_2}. \tag{9}$$

For heterogeneous networks, $\lambda_2 \ll \lambda_{\max}$. Thus, $\tau < \tau_c = \frac{1}{c\lambda_{\max}} \ll \frac{1}{c\lambda_2}$, that is, $c\tau\lambda_2 \ll 1$. From Eq. (9), one has $\mu(\tau) \approx (1 + c\tau\lambda_2)\mu(0)$. It shows that the time-delay τ $(\tau < \tau_c)$ could increase the converging speed linearly for heterogeneous networks. While for homogeneous networks, $\lambda_{\max}$ could be a little larger than λ_2. Particularly, for some regular homogeneous networks (e.g., complete networks), $\lambda_{\max}$ equals λ_2. Thus, $1 - c\tau\lambda_2$ could be very small while $\tau < \tau_c$. From Eq. (9), one can find that the time delay could increase the converging speed significantly for homogeneous networks.

In order to test our main results of critical time-delay tolerance and the converging speed without or with time de-

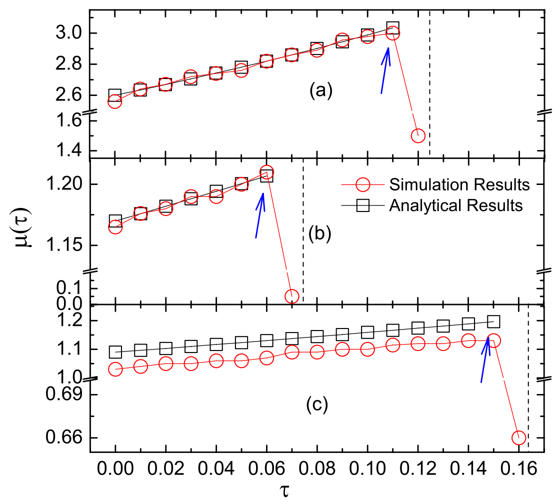

FIG. 2. (Color online) Converging speed $\mu(\tau)$ as a function of the value of time-delay τ for different networks: (a) $\langle k\rangle=40$, $\delta=0.0$; (b) $\langle k\rangle=20$, $\delta=1.0$; (c) $\langle k\rangle=20$, $\delta=0.0$. The second least eigenvalues and the largest eigenvalues of the three networks are (a) $\lambda_2=16.4$, $\lambda_{\max}=124.0$, (b) $\lambda_2=7.3$, $\lambda_{\max}=192.4$, and (c) $\lambda_2=6.8$, $\lambda_{\max}=81.3$, respectively. The network size $N=1500$ and the coupling strength $c=0.08$. The curves of (○) represent simulation results while the curves of (□) represent the analytical results given by Eq. (7). The arrows point to the critical time-delay tolerance τ_c and the dashed lines point out the beginning of diverging region (see the text for details). The simulation results are obtained by averaging over 10 different generations of the network model.

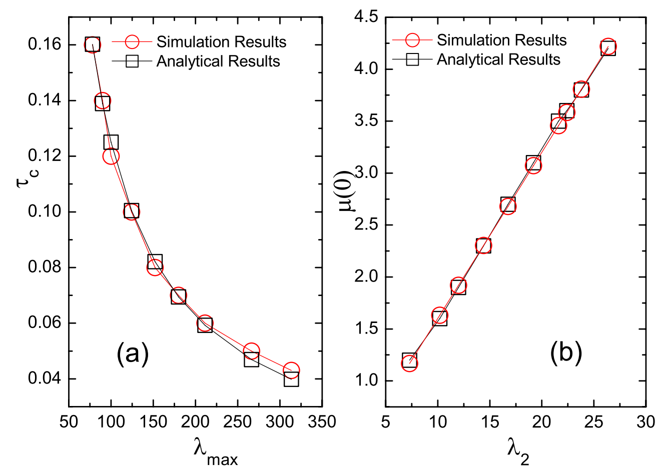

FIG. 3. (Color online) (a) The critical time-delay tolerance τ_c vs the largest eigenvalue $\lambda_{\max}$. (b) Converging speed $\mu(0)$ without time-delay vs the second least eigenvalue λ_2. The curves of (○) represent simulation results while the curves of (□) represent analytical results. We fixed the network size $N=1500$ and generated different networks with various λ_2 and $\lambda_{\max}$ by tuning the values of δ and m in the network model (see the text for details). The coupling strength $c=0.08$. The simulation results are obtained by averaging over 10 different generations of the network model.

lay, represented by Eqs. (4), (8), and (9), respectively, we have performed extensive numerical experiments. As we need to generate various networks with different values of λ_2 and λ_N, we adopt the evolution network model in Refs. [36,37]. In the initial network, there are m_0 connected nodes. Then, add one node into the network at each time step and connect it to m ($m\leq m_0$) existing nodes. The node links to the existing node i with probability $\Pi(k_i)=k_i^{\delta}/\Sigma_j k_j^{\delta}$, where k_i is the degree of node i and δ is a constant describing nonlinear preferential attachment. After a sufficiently long time evolution (or for a large enough size, since $N=m_0+t$), the average degree of the whole network is $\langle k\rangle\simeq 2$ m and the node degrees distribution could be in a power-law form if $\delta=1$, or else otherwise. Some previous studies have indicated that the average degree $\langle k\rangle$ and node-degree heterogeneity are related to the values of λ_2 and λ_N [38]. To reproduce that, we generated various networks needed by using this model by tuning the values of m and δ.

As an example, consider a network of coupled Rössler oscillators, i.e., $\mathbf{x}_i=(x_{i1},x_{i2},x_{i3})^T$ and $\mathbf{F}(\mathbf{x}_i)=[-x_{i2}-x_{i3};x_{i1}+0.2x_{i2};0.2+(x_{i1}-7.0)x_{i3}]$. According to the theoretical analysis above, we computed an order parameter used to measure the synchronization error: $W(t)=\Sigma_i|\mathbf{x}_i-\langle\mathbf{x}_i\rangle|^2$ where $\langle\dots\rangle$ denotes averaging over all nodes. It is obvious that for large t, $W(t)\to 0$, if the network can be synchronized. And the converging speed $\mu(\tau)=-\frac{d\ln W(t)}{dt}$, which indicates that we can obtain converging speed by computing the absolute value of the slope of the evolution curve of $W(t)$ in semilog plot as in Fig. 1(c). In the numerical simulations, we used the Euler method with time step $\Delta t=0.01$. Figure 1(c) displays the typical evolution curves of $W(t)$ for four different networks: from right to left, (I) $\langle k\rangle=20$, $\delta=0.0$, $\tau=0$, (II) $\langle k\rangle=30$, $\delta=0.5$, $\tau=0$, (III)$\langle k\rangle=40$, $\delta=1.0$, $\tau=0$, and (IV)$\langle k\rangle=40$, $\delta=1.0$, $\tau=0.05$. Comparing (I) and (III), one can easily find that larger average degree induces faster convergence while comparing (III) and (IV) one finds that proper time delay could increase the converging speed as implied by Eq. (9). Figures 1(a) and 1(b) display the function $\zeta_h(t)$ in Eq. (6) for $h=2$ and $h=N$, respectively, on the network with $N=1500$, $\langle k\rangle=40$, and $\delta=1.0$, the time-delay $\tau=0.05$. One can see that the values are indeed fluctuated around zero, which indicates that the assumption in our theoretical analysis is valid.

Next, we show extensive simulations to verify the accuracy of our main results. By adding some time-delay τ into the networks, we obtained the results for three heterogeneous networks as shown in Fig. 2. One can find that the converging speed increases almost linearly with the value of τ ($<\tau_c$), which is in accordance with our analytical result. It is noticeable that when the time delay is less than τ_c the converging speed increases with the value of the time delay according to Eq. (9), however, in simulations when the time delay is a little larger than τ_c the system may not diverge but instead its converging speed decreases quickly (within a small interval). And after that, the system will diverge, i.e., the oscillator network will not become synchronized. We point out the beginning of the diverging region by dashed lines in Fig. 2. Therefore, we can get the value of the critical time-delay tolerance τ_c at the turning point in simulations (as pointed by arrows in the figure). Furthermore, we tuned the values of δ and m in the network generating model and thus get various networks with different values of λ_2 and λ_N. Then, we performed the numerical simulations on these networks to get the relation between critical time-delay toler-

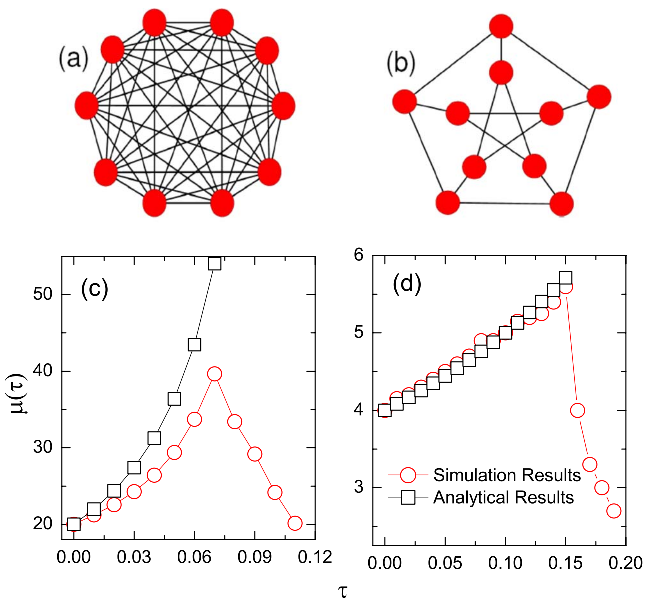

FIG. 4. (Color online) (c) and (d) display the simulation results of the converging speed $\mu(\tau)$ as a function of τ for the complete network (a) and the Cage network (b), respectively. The network size $N=10$ and the coupling strength $c=1.0$.

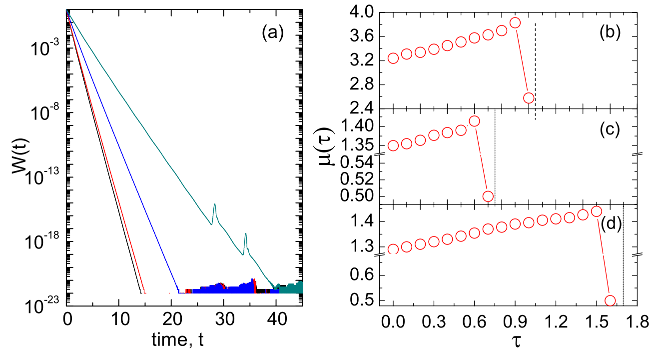

FIG. 5. (Color online) The results for the coupling function $\mathbf{H}(\mathbf{x})=(x_1,x_2,0)$. (a) displays $W(t)$ as a function of the evolution time t for different networks with the same size $N=1500$. The other parameters are the same as that in Fig. 1. And (b), (c), and (d) display $\mu(\tau)$ as a function of the value of τ for three networks with the same parameters as in Fig. 2. The coupling strength $c=0.1$. The curves in (b), (c), and (d) present the averaging results over 10 different generations of the network model and the dashed lines point out the beginning of diverging region.

ance τ_c and the largest eigenvalue $\lambda_{\max}$, between converging speed $\mu(0)$ without time delay and the second least eigenvalue λ_2. The results are exhibited in Fig. 3. One can find that in Fig. 3(a) the critical time-delay tolerance decays as the maximal eigenvalues increase, which is in accordance with our theoretical result described by Eq. (3). And Fig. 3(b) show that $\mu(0)$ is proportional to λ_2, which is also in accordance with our analytical result described by Eq. (8). Note that Refs. [39,40] studied synchronization of pulse-coupled biological oscillators on random networks and Kuramoto oscillators on hierachical networks, respectively. And their results showed that the synchronizing time is inversely proportional to λ_2, which supports our analytical result about converging speed without time-delay $\mu(0)$ as described by Eq. (8). At last, we choose two small-size networks ($N=10$) to show the different effect of time delay for homogeneous and heterogeneous networks. For the complete network ($\lambda_2=\lambda_N=10$), Fig. 4(a) shows that $\mu(\tau)$ increase quickly as τ. While for the Cage network [25] ($\lambda_2=2$ and $\lambda_N=5$), Fig. 4(b) shows that the value of $\mu(\tau)$ increase linearly. It is worth noting that according to Eq. (9), the converging speed will approaching infinite when the time delay is close to the critical time-delay τ_c for the complete networks, while the simulation result in Fig. 4(c) shows that the converging speed will decrease before the critical time-delay τ_c. However, by comparing Fig. 4(c) to Fig. 4(d) one can find that the converging speed for the complete network increases much more quickly than that for the Cage network. The comparative result is in accordance with that in Eq. (9).

The theoretical and numerical results above are all obtained for the coupling function $\mathbf{H}(\mathbf{x})=\mathbf{x}$. In that case, the synchronizable region is unbounded, i.e., $\alpha_b=\infty$. It is noted that some basic analytic results for general time-delayed network synchronization were developed in [41], but the approach taken in this paper is more transparent to the underlying ideas and is consistent with Ref. [41].

Next, we perform numerical simulations for another coupling function, with $\alpha_b<\infty$. It is noteworthy that in Ref. [28] the authors pointed that the unweighted heterogeneous networks cannot be synchronized for $\mathbf{H}(\mathbf{x})=(x_1,0,0)$ if the network size $N\gtrsim 1000$. As we need to study the synchronizing process on various networks of large sizes, we choose the coupling function $\mathbf{H}(\mathbf{x})=(x_1,x_2,0)$ for which the value of α_b is also finite. The results are shown in Fig. 5. One can see that for different networks the converging speeds are different, and proper time delays can increase converging speeds as shown in Figs. 5(b)–5(d). General results are almost the same as that for the coupling function $\mathbf{H}(\mathbf{x})=\mathbf{x}$ although the values have some small differences.

In conclusion, we have studied both theoretically and numerically the performances of synchronizing process of some complex oscillator networks, which revealed the relation between the performance and the network topology. We found that the critical time-delay tolerance is inversely proportional to the largest Laplacian eigenvalue of the network and the converging speed without time delay is in proportional to the second least eigenvalue. Furthermore, we found that the time delay could increase the converging speed linearly for heterogeneous networks and significantly for homogeneous networks. Since time-delay tolerance and converging speed are important in real-world problems, our results should be useful for structure design and technological application issues of network synchronization.

We acknowledge the useful discussions with Jie Ren and the financial support from NNSFC under Grant Nos. 70671097, 60974079, 60772158, and 60221301, the 973 project of China under Grant No. 2007CB310805, the Important Direction Item of Knowledge Innovation Project of CAS under Grant No. KJCX3-SYW-S01, NNSFC-HKRGC Joint Research Scheme under Grant No. N-CityU 107/07.

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